AP Calculus AB — Homework 0213 - Solutions
Dr. Paul L. Bailey — Thursday, February 13, 2025

Problem 1 (Thomas §4.2 # 23)**.** Suppose that $f(1) = 3$ and that $f'(x) = 0$ for all $x \in (0, 2)$. Must $f(x) = 3$ for all $x \in (0, 2)$? for all $x \in [0, 2]$? Give reasons for your answer.

Solution. For all $x \in (0, 2)$, yes. To see how to write the reasoning, look at the next problem.

For all $x \in [0, 2]$, no. For example:

$$f(x) = \begin{cases} 3 & \text{if } x \in [0, 2) \\ 4 & \text{if } x = 2 \end{cases}$$

This satisfies the hypothesis of the problem but not the conclusion. □

Problem 2 (Thomas §4.2 # 24)**.** Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all $x \in (-2, 2)$. Must $f(x) = 2x + 5$ for all $x \in (-2, 2)$? Give reasons for your answer.

Solution. Yes. Let $g(x) = 2x$. Suppose that $x_0 \in (0, 2)$ and $x_1 \in (x_0, 2)$. Then f and g are continuous on $[0, x_1]$ and differentiable on $(0, x_1)$, with $f'(x) = g'(x) = 2$. So, by MVT Corollary 2, $f(x) = g(x) + C$ for some $C \in \mathbb{R}$. Thus $5 = f(0) = g(0) + C = 0 + C = C$, so $C = 5$, and $f(x) = 2x + 5$. □

Problem 3 (Thomas §4.2 # 27)**.** Find all possible functions with the given derivative.

(a) x

(b) x^2

(c) x^3

Solution. The answers are the antiderivatives

(a) $\dfrac{x^2}{2} + C$

(b) $\dfrac{x^3}{3} + C$

(c) $\dfrac{x^4}{4} + C$

□

Problem 4 (Thomas §4.2 # 28)**.** Find all possible functions with the given derivative.

(a) $2x$

(b) $2x - 1$

(c) $3x^2 + 2x - 1$

Solution. The answers are the antiderivatives

(a) $x^2 + C$

(b) $x^2 - x + C$

(c) $x^3 + x^2 - x + C$

□

Problem 5 (Thomas §4.2 # 41). A body moves with acceleration $a = d^2s/dt^2$, initial velocity $v(0)$, and initial position $s(0)$ along a coordinate line, where

$$a = 32, v_0 = 20, s_0 = 5$$

Find the body's position at time t.

Solution. We have $s''(t) = a = 32$, so $s'(t) = 32t + 20$, so $s(t) = 16t^2 + 20t + 5$. □

Problem 6. Compute dy/dx. Simplify.

(a) $y = \dfrac{x^2 - 4}{x - 2}$

(b) $y = \dfrac{x^2 + 3x - 1}{x - 2}$

(c) $y = \sec^2 x - \tan^2 x$

Solution. Manipulate y first.

(a) Factor the top, then cancel.

$$y = \frac{x^2 - 4}{x - 2} = \frac{(x+2)(x-2)}{x-2} = x + 2 \quad \Rightarrow \quad \frac{dy}{dx} = 1$$

(b) Break it up, factor, and cancel.

$$y = \frac{x^2 + 3x - 1}{x - 2} = \frac{x^2 + 3x - 10}{x - 2} + \frac{9}{x - 2} = x + 5 + \frac{9}{x - 2} \quad \Rightarrow \quad \frac{dy}{dx} = 1 - \frac{9}{(x-2)^2}$$

(c) Use a trig identity.

$$y = \sec^2 x - \tan^2 x = (1 + \tan^2 x) - \tan^2 x = 1 \quad \Rightarrow \quad \frac{dy}{dx} = 0$$

□

Problem 7. Let
$$f(x) = \frac{x^2 - 15}{x - 4}.$$

(a) Solve $f'(x) = 0$.

(b) Find the domain and range of f.

(c) The graph of f has two linear asymptotes. Write the equations for these lines.

Solution. Use the quotient rule to compute that
$$f'(x) = \frac{2x(x-4) - (x^2 - 15)}{(x-4)^2} = \frac{x^2 - 8x + 15}{(x-4)^2} = \frac{(x-3)(x-5)}{(x-4)^2}.$$
If $f'(x) = 0$, then $x = 3$ or $x = 5$. These give the local min and max, respectively.

The domain of f is $\mathbb{R} \setminus \{4\}$. Since $f(3) = 6$ and $f(5) = 10$, the range is $(-\infty, 3] \cup [5, \infty)$.

Since f has a pole at $x = 4$, the line $x = 4$ is a vertical asymptote. The slant asymptote is given by dividing the bottom into the top to find that
$$f(x) = x + 4 + \frac{1}{x-4}.$$
Since $\lim_{x\to\infty} \dfrac{1}{x-4} = 0$, f has a slant asymptote at $y = x + 4$. □

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin(1 + x^2)$. Find all $x \in \mathbb{R}$ such that f is differentiable at x.

Solution. Since $f'(x) = 2x\cos(1 + x^2)$ for all $x \in \mathbb{R}$, f is differentiable on $\mathbb{R}$. □

Problem 9 (Thomas Ch 2 Practice # 15)**.** Find
$$\lim_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{x}}{x}.$$

Solution. We have
$$\lim_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{x}}{x} = \lim_{x\to 0} \frac{-2}{x^2(2+x)} = -\infty.$$
However, this is a typo. The problem in the book is to find
$$\lim_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}.$$
Here we have
$$\lim_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{x}}{x} = \lim_{x\to 0} \frac{x}{2x(2+x)} = \lim_{x\to 0} \frac{1}{2(2+x)} = \frac{1}{4}.$$
□

Problem 10 (Thomas §2.6 # 47)**.** Show that the equation
$$x^3 - 15x + 1 = 0$$
has three solutions in the interval $[-4, 4]$. (Hint: use IVT.)

Solution. We have $f(-4) = -3$, $f(0) = 1$, $f(1) = -13$, and $f(4) = 5$. Thus f has a zero in each of the intervals $(-4, 0)$, $(0, 1)$, and $(1, 4)$. Since it is cubic, it has at most three zeros; thus it has exactly three. □